

Higher Auslander Algebras Admitting Trivial Maximal Orthogonal Subcategories ^{*†}

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Abstract

For an Artinian $(n-1)$ -Auslander algebra Λ with global dimension $n(\geq 2)$, we show that if Λ admits a trivial maximal $(n-1)$ -orthogonal subcategory of $\text{mod } \Lambda$, then Λ is a Nakayama algebra and the projective or injective dimension of any indecomposable module in $\text{mod } \Lambda$ is at most $n-1$. As a result, for an Artinian Auslander algebra with global dimension 2, if Λ admits a trivial maximal 1-orthogonal subcategory of $\text{mod } \Lambda$, then Λ is a tilted algebra of finite representation type. Further, for a finite-dimensional algebra Λ over an algebraically closed field K , we show that Λ is a basic and connected $(n-1)$ -Auslander algebra Λ with global dimension $n(\geq 2)$ admitting a trivial maximal $(n-1)$ -orthogonal subcategory of $\text{mod } \Lambda$ if and only if Λ is given by the quiver:

$$1 \xleftarrow{\beta_1} 2 \xleftarrow{\beta_2} 3 \xleftarrow{\beta_3} \cdots \xleftarrow{\beta_n} n+1$$

modulo the ideal generated by $\{\beta_i\beta_{i+1} | 1 \leq i \leq n-1\}$. As a consequence, we get that a finite-dimensional algebra over an algebraically closed field K is an $(n-1)$ -Auslander algebra with global dimension $n(\geq 2)$ admitting a trivial maximal $(n-1)$ -orthogonal subcategory if and only if it is a finite direct product of K and Λ as above. Moreover, we give some necessary condition for an Artinian Auslander algebra admitting a non-trivial maximal 1-orthogonal subcategory.

1. Introduction

It is well known that the notion of maximal n -orthogonal subcategories introduced by Iyama in [Iy3] played a crucial role in developing the higher-dimensional Auslander-Reiten theory (see [Iy3] and [Iy4]). This notion coincides with that of $(n+1)$ -cluster tilting subcategories introduced by Keller and Reiten in [KR]. In general, maximal n -orthogonal subcategories rarely exist. So it would be interesting to investigate when maximal n -orthogonal subcategories exist and the properties of algebras admitting such subcategories. Several authors have worked on this topics (see [EH], [GLS], [HuZ], [Iy3], [Iy4], [Iy5], [Iy6], [L], and so

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on). As a generalization of the notion of the classical Auslander algebras, Iyama introduced the notion of n -Auslander algebras in [Iy5]. Then he proved that for a finite-dimensional $(n-1)$ -Auslander algebra Λ with global dimension $n(\geq 2)$ over an algebraically closed field K , Λ has maximal $(n-1)$ -orthogonal modules in $\text{mod } \Lambda$ if and only if Λ is Morita equivalent to $T_m^{(n)}(K)$ for some $m \geq 1$, where $T_m^{(1)}(K)$ is an $m \times m$ upper triangular algebra and $T_m^{(n)}(K)$ is the endomorphism algebra of a maximal $(n-2)$ -orthogonal module in $\text{mod } T_m^{(n-1)}(K)$. Moreover, he gave some examples of the quivers of these algebras inductively. In [HuZ] we proved that an Artinian $(n-1)$ -Auslander algebra Λ with global dimension $n(\geq 2)$ admits a trivial maximal $(n-1)$ -orthogonal subcategory of $\text{mod } \Lambda$ if and only if any simple module $S \in \text{mod } \Lambda$ with projective dimension n is injective. In [HuZ] we also proved that for an almost hereditary algebra Λ with global dimension 2, if Λ admits a maximal 1-orthogonal subcategory \mathcal{C} of $\text{mod } \Lambda$, then \mathcal{C} is trivial. In this paper, we continue to study the structure of an $(n-1)$ -Auslander algebra Λ admitting a trivial maximal $(n-1)$ -orthogonal subcategory of $\text{mod } \Lambda$. This paper is organized as follows.

In Section 2, we give some notions and notations and collect some preliminary results about minimal morphisms. In Section 3, we give some homological properties of indecomposable modules (in particular, simple modules) over higher Auslander algebras (admitting a trivial maximal orthogonal subcategory of $\text{mod } \Lambda$).

Let Λ be an Artinian $(n-1)$ -Auslander algebra Λ with global dimension $n(\geq 2)$ admitting a trivial maximal $(n-1)$ -orthogonal subcategory of $\text{mod } \Lambda$. We get in Section 4 the following conclusions: (1) For any simple module $S \in \text{mod } \Lambda$ with $\text{pd}_\Lambda S = n$, the i th syzygy module of S is simple $1 \leq i \leq n$ and all terms in a minimal projective resolution of S are indecomposable. (2) The sum of the projective and injective dimensions of any non-projective-injective indecomposable module in $\text{mod } \Lambda$ is equal to n . As a consequence, the projective or injective dimension of any indecomposable module in $\text{mod } \Lambda$ is at most $n-1$. (3) Λ is a Nakayama algebra. As a result, we get that an Artinian Auslander algebra Λ admitting a trivial maximal 1-orthogonal subcategory of $\text{mod } \Lambda$ is a tilted algebra of finite representation type.

As an application of the results obtained in Section 4, we give in Section 5 an explicit description of the quiver of a finite-dimensional $(n-1)$ -Auslander algebra Λ with global dimension n admitting a trivial maximal $(n-1)$ -orthogonal subcategory of $\text{mod } \Lambda$. Let K be an algebraically closed field. We first prove that for a basic and connected finite-dimensional $(n-1)$ -Auslander K -algebra Λ , if Λ admits a trivial maximal $(n-1)$ -orthogonal subcategory of $\text{mod } \Lambda$, then there exists a unique simple module $S \in \text{mod } \Lambda$ with projective

dimension n . Then we prove that Λ is a basic and connected finite-dimensional $(n - 1)$ -Auslander K -algebra admitting a trivial maximal $(n - 1)$ -orthogonal subcategory of $\text{mod } \Lambda$ if and only if Λ is given by the quiver:

$$1 \xleftarrow{\beta_1} 2 \xleftarrow{\beta_2} 3 \xleftarrow{\beta_3} \cdots \xleftarrow{\beta_n} n + 1$$

modulo the ideal generated by $\{\beta_i/\beta_{i+1} | 1 \leq i \leq n - 1\}$. As a consequence, we establish the following structure theorem: a finite-dimensional $(n - 1)$ -Auslander K -algebra with global dimension $n(\geq 2)$ admitting a trivial maximal $(n - 1)$ -orthogonal subcategory of $\text{mod } \Lambda$ if and only if it is a finite direct product of K and Λ as above.

By [Iy5], there exist an Auslander algebra with global dimension 2 admitting a non-trivial maximal 1-orthogonal subcategory. On the other hand, by [HuZ, Corollary 3.12] we have that if Λ is an Artinian Auslander algebra with global dimension 2 admitting a non-trivial maximal 1-orthogonal subcategory of $\text{mod } \Lambda$, then there exists a simple module $S \in \text{mod } \Lambda$ such that both the projective and injective dimensions of S are equal to 2. In Section 6, we further give a necessary condition for an Auslander algebra with global dimension 2 admitting a non-trivial maximal 1-orthogonal subcategory in terms of the homological properties of simple modules. We prove that if Λ is an Artinian Auslander algebra with global dimension 2 admitting a non-trivial maximal 1-orthogonal subcategory of $\text{mod } \Lambda$, then there exist at least two non-injective simple modules in $\text{mod } \Lambda$ with projective dimension 2.

2. The properties of minimal morphisms

In this section, we give some notions and notations in our terminology and collect some preliminary results about minimal morphisms for later use.

Throughout this paper, Λ is an Artinian algebra with the center R , $\text{mod } \Lambda$ is the category of finitely generated left Λ -modules and $\text{gl.dim } \Lambda$ denotes the global dimension of Λ . We denote by \mathbb{D} the ordinary duality, that is, $\mathbb{D}(-) = \text{Hom}_R(-, I(R/J(R)))$, where $J(R)$ is the Jacobson radical of R and $I(R/J(R))$ is the injective envelope of $R/J(R)$.

Let M be in $\text{mod } \Lambda$. We use

$$\cdots \rightarrow P_i(M) \rightarrow \cdots \rightarrow P_1(M) \rightarrow P_0(M) \rightarrow M \rightarrow 0$$

and

$$0 \rightarrow M \rightarrow I^0(M) \rightarrow I^1(M) \rightarrow \cdots \rightarrow I^i(M) \rightarrow \cdots$$

to denote the minimal projective resolution and the minimal injective resolution of M , respectively. In particular, $P_0(M)$ and $I^0(M)$ are the projective cover and the injective envelope of M , respectively. Denote by $\Omega^i M$ and $\Omega^{-i} M$ the i th syzygy and co-syzygy of M , respectively.

The following easy observations are well-known.

Lemma 2.1 *Let $M \in \text{mod } \Lambda$ and $M \cong M_1 \oplus M_2$. Then*

$$0 \rightarrow M(\cong M_1 \oplus M_2) \rightarrow I^0(M') \oplus I^0(M'') \rightarrow I^1(M') \oplus I^1(M'') \rightarrow \dots$$

and

$$\dots \rightarrow P_1(M') \oplus P_1(M'') \rightarrow P_0(M') \oplus P_0(M'') \rightarrow M(\cong M_1 \oplus M_2) \rightarrow 0$$

are a minimal injective resolution and a minimal projective resolution of M , respectively, and $\Omega^{-i}M \cong \Omega^{-i}M_1 \oplus \Omega^{-i}M_2$ and $\Omega^iM \cong \Omega^iM_1 \oplus \Omega^iM_2$ for any $i \geq 1$.

Lemma 2.2 *Let M and S be in $\text{mod } \Lambda$ with S simple. Then $\text{Ext}_\Lambda^i(S, M) \cong \text{Hom}_\Lambda(S, \Omega^{-i}M)$ for any $i \geq 0$.*

Recall from [AuR] that a morphism $f : M \rightarrow N$ in $\text{mod } \Lambda$ is said to be *left minimal* if an endomorphism $g : N \rightarrow N$ is an automorphism whenever $f = gf$. Dually, the notion of *right minimal* morphisms is defined.

Lemma 2.3 ([Au, Chapter II, Lemma 4.3]) *Let $0 \rightarrow A \xrightarrow{g} B \xrightarrow{f} C \rightarrow 0$ be a non-split exact sequence in $\text{mod } \Lambda$.*

- (1) *If A is indecomposable, then $f : B \rightarrow C$ is right minimal.*
- (2) *If C is indecomposable, then $g : A \rightarrow B$ is left minimal.*

By Lemma 2.3, we immediately have the following result.

Corollary 2.4 *Let $M \in \text{mod } \Lambda$ be an indecomposable non-injective module and $I^0(M)$ projective. Then*

$$\dots \rightarrow P_i(M) \rightarrow \dots \rightarrow P_1(M) \rightarrow P_0(M) \rightarrow I^0(M) \xrightarrow{\pi} I^0(M)/M \rightarrow 0$$

is a minimal projective resolution of $I^0(M)/M$, where π is the natural epimorphism.

The following properties of minimal morphisms are useful in the rest of the paper.

Lemma 2.5 *Let $0 \rightarrow A \xrightarrow{g} B \xrightarrow{f} C \rightarrow 0$ be a non-split exact sequence in $\text{mod } \Lambda$.*

- (1) *If g is left minimal, then $\text{Ext}_\Lambda^1(C', A) \neq 0$ for any non-zero direct summand C' of C .*
- (2) *If f is right minimal, then $\text{Ext}_\Lambda^1(C, A') \neq 0$ for any non-zero direct summand A' of A .*

Proof. (1) If $\text{Ext}_\Lambda^1(C', A) = 0$ holds for some non-zero direct summand C' of C . Then we have the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & A & \longrightarrow & C' \oplus A & \xrightleftharpoons[\substack{\pi_3 \\ i_3}]{} & C' \longrightarrow 0 \\
& & \parallel & & \downarrow i_1 & & \downarrow \pi_2 \\
0 & \longrightarrow & A & \xrightarrow{g} & B & \xrightarrow{f} & C \longrightarrow 0
\end{array}$$

$\downarrow i_2$

such that $\pi_3 i_3 = 1_{C'} = \pi_2 i_2$ and $i_2 \pi_3 = f i_1$. Then $1_{C'} = (\pi_2 i_2)(\pi_3 i_3) = (\pi_2 f)(i_1 i_3)$, and hence C' is a direct summand of B and $(\pi_2 f)g = 0$. By [AuRS, Chapter I, Theorem 2.4], g is not left minimal, which is a contradiction.

Similarly, we get (2). □

The following lemma establishes a connection between left minimal morphisms and right minimal morphisms.

Lemma 2.6 *Let*

$$0 \rightarrow A \xrightarrow{g} B \xrightarrow{f} C \rightarrow 0 \quad (1)$$

be a non-split exact sequence in $\text{mod } \Lambda$ with B projective-injective. Then the following are equivalent.

- (1) *A is indecomposable and g is left minimal.*
- (2) *C is indecomposable and f is right minimal.*

Proof. (1) \Rightarrow (2) Since A is indecomposable, f is right minimal by Lemma 2.3. Then B is projective implies that the exact sequence (1) is part of a minimal projective resolution of C . If $C = C_1 \oplus C_2$ with C_1 and C_2 non-zero, then neither C_1 nor C_2 are projective by Lemma 2.5. So both $\Omega^1 C_1$ and $\Omega^1 C_2$ are non-zero and $A \cong \Omega^1 C_1 \oplus \Omega^1 C_2$, which contradicts with that A is indecomposable.

Similarly, we get (2) \Rightarrow (1). □

3. Higher Auslander algebras and maximal orthogonal subcategories

In this section, we give the definitions of higher Auslander algebras and maximal orthogonal subcategories, which were introduced by Iyama in [Iy5] and [Iy3], respectively. Then we study the homological behavior of indecomposable modules (in particular, simple modules) over higher Auslander algebras (admitting a trivial maximal orthogonal subcategory of $\text{mod } \Lambda$).

As a generalization of the notion of classical Auslander algebras, Iyama introduced in [Iy5] the notion of n -Auslander algebras as follows.

Definition 3.1 ([Iy5]) For a positive integer n , Λ is called an n -Auslander algebra if $\text{gl.dim } \Lambda \leq n + 1$ and $I^0(\Lambda), I^1(\Lambda), \dots, I^n(\Lambda)$ are projective.

The notion of n -Auslander algebras is left and right symmetric by [Iy5, Theorem 1.10]. It is trivial that n -Auslander algebras with global dimension at most n are semisimple. In particular, the notion of 1-Auslander algebras is just that of classical Auslander algebras. In the following, we assume that $n \geq 2$ when an $(n-1)$ -Auslander algebra is concerned.

Denote by $\mathcal{P}\mathcal{I}^n(\Lambda)$ (resp. $\mathcal{I}\mathcal{P}^n(\Lambda)$) the subcategory of $\text{mod } \Lambda$ consisting of indecomposable projective modules with injective dimension n (resp. indecomposable injective modules with projective dimension n). By applying Lemma 2.6 to $(n-1)$ -Auslander algebras, we get the following result.

Lemma 3.2 *Let Λ be an $(n-1)$ -Auslander algebra with $\text{gl.dim } \Lambda = n$. Then we have the following*

(1) *For any $P \in \mathcal{P}\mathcal{I}^n(\Lambda)$, the minimal injective resolution of P :*

$$0 \rightarrow P \rightarrow I^0(P) \rightarrow I^1(P) \rightarrow \cdots \rightarrow I^n(P) \rightarrow 0 \quad (2)$$

is a minimal projective resolution of $I^n(P)$ and $I^n(P)$ is indecomposable.

(2) *For any module $I \in \mathcal{I}\mathcal{P}^n(\Lambda)$, the minimal projective resolution of I :*

$$0 \rightarrow P_n(I) \rightarrow \cdots \rightarrow P_1(I) \rightarrow P_0(I) \rightarrow I \rightarrow 0$$

is a minimal injective resolution of $P_n(I)$ and $P_n(I)$ is indecomposable.

Proof. (1) Since Λ is an $(n-1)$ -Auslander algebra, by Lemma 2.1, it is easy to see that $I^i(P)$ is projective for any $0 \leq i \leq n-1$. So the exact sequence (2) is a projective resolution of $I^n(P)$, and then the assertion follows from Lemma 2.6.

Dually, we get (2). □

By Lemma 3.2, we get immediately the following result.

Lemma 3.3 *Let Λ be an $(n-1)$ -Auslander algebra with $\text{gl.dim } \Lambda = n$. Then Ω^n gives a one-one correspondence between $\mathcal{I}\mathcal{P}^n(\Lambda)$ and $\mathcal{P}\mathcal{I}^n(\Lambda)$ with the inverse Ω^{-n} .*

For a module $M \in \text{mod } \Lambda$, we use $\text{pd}_\Lambda M$ and $\text{id}_\Lambda M$ to denote the projective dimension and the injective dimension of M , respectively.

Lemma 3.4 *Let Λ be an $(n-1)$ -Auslander algebra with $\text{gl.dim } \Lambda = n$ and $S \in \text{mod } \Lambda$ a simple module with $\text{pd}_\Lambda S = n$, then $P_n(S)$ is indecomposable.*

Proof. Let Λ be an $(n-1)$ -Auslander algebra with $\text{gl.dim } \Lambda = n$ and $S \in \text{mod } \Lambda$ a simple module with $\text{pd}_\Lambda S = n$. By [Iy2, Proposition 6.3(2)], $\text{Ext}_\Lambda^n(S, \Lambda) \in \text{mod } \Lambda^{op}$ is simple. By [HuZ, Lemma 2.4], $S \not\subseteq I^0(\Lambda) \oplus \cdots \oplus I^{n-1}(\Lambda)$. So $\text{Ext}_\Lambda^i(S, \Lambda) \cong \text{Hom}_\Lambda(S, I^i(\Lambda)) = 0$ for

any $0 \leq i \leq n-1$ by Lemma 2.2. Then from the minimal projective resolution of S , we get the exact sequence:

$$0 \rightarrow P_0(S)^* \rightarrow \cdots \rightarrow P_{n-1}(S)^* \rightarrow P_n(S)^* \rightarrow \text{Ext}_\Lambda^n(S, \Lambda) \rightarrow 0$$

which is a minimal projective resolution of $\text{Ext}_\Lambda^n(S, \Lambda)$ by [M, Proposition 4.2], where $(-)^* = \text{Hom}_\Lambda(-, \Lambda)$. So $P_n(S)^* \cong P_0(\text{Ext}_\Lambda^n(S, \Lambda))$ is indecomposable and hence $P_n(S)$ is also indecomposable. \square

Denote by $\mathcal{P}^n(S)$ and $\mathcal{I}^n(S)$ the subcategory of $\text{mod } \Lambda$ consisting of simple modules with projective dimension n and injective dimension n , respectively. Since \mathbb{D} is a duality between simple Λ -modules and simple Λ^{op} -modules, we get easily the following result from [Iy2, Proposition 6.3].

Lemma 3.5 *Let Λ be an $(n-1)$ -Auslander algebra with $\text{gl.dim } \Lambda = n$. Then the functor $\mathbb{D} \circ \text{Ext}_\Lambda^n(-, \Lambda)$ gives a bijection from $\mathcal{P}^n(S)$ to $\mathcal{I}^n(S)$ with the inverse $\text{Ext}_\Lambda^n(-, \Lambda) \circ \mathbb{D}$.*

Let \mathcal{C} be a full subcategory of $\text{mod } \Lambda$ and n a positive integer. Recall from [AuR] that \mathcal{C} is said to be *contravariantly finite* in $\text{mod } \Lambda$ if for any $M \in \text{mod } \Lambda$, there exists a morphism $C_M \rightarrow M$ with $C_M \in \mathcal{C}$ such that $\text{Hom}_\Lambda(C, C_M) \rightarrow \text{Hom}_\Lambda(C, M) \rightarrow 0$ is exact for any $C \in \mathcal{C}$. Dually, the notion of *covariantly finite subcategories* of $\text{mod } \Lambda$ is defined. A full subcategory of $\text{mod } \Lambda$ is said to be *functorially finite* in $\text{mod } \Lambda$ if it is both contravariantly finite and covariantly finite in $\text{mod } \Lambda$. We denote by ${}^{\perp n}\mathcal{C} = \{X \in \text{mod } \Lambda \mid \text{Ext}_\Lambda^i(X, C) = 0 \text{ for any } C \in \mathcal{C} \text{ and } 1 \leq i \leq n\}$, and $\mathcal{C}^{\perp n} = \{X \in \text{mod } \Lambda \mid \text{Ext}_\Lambda^i(C, X) = 0 \text{ for any } C \in \mathcal{C} \text{ and } 1 \leq i \leq n\}$.

Definition 3.6 ([Iy3]) Let \mathcal{C} be a functorially finite subcategory of $\text{mod } \Lambda$. For a positive integer n , \mathcal{C} is called a *maximal n -orthogonal subcategory* of $\text{mod } \Lambda$ if $\mathcal{C} = {}^{\perp n}\mathcal{C} = \mathcal{C}^{\perp n}$.

From the definition above, we get easily that both Λ and $\mathbb{D}\Lambda^{op}$ are in any maximal n -orthogonal subcategory of $\text{mod } \Lambda$. For a module $M \in \text{mod } \Lambda$, we use $\text{add}_\Lambda M$ to denote the subcategory of $\text{mod } \Lambda$ consisting of all modules isomorphic to direct summands of finite direct sums of copies of ${}_\Lambda M$. Then $\text{add}_\Lambda(\Lambda \oplus \mathbb{D}\Lambda^{op})$ is contained in any maximal n -orthogonal subcategory of $\text{mod } \Lambda$. On the other hand, it is easy to see that if $\text{add}_\Lambda(\Lambda \oplus \mathbb{D}\Lambda^{op})$ is a maximal n -orthogonal subcategory of $\text{mod } \Lambda$, then $\text{add}_\Lambda(\Lambda \oplus \mathbb{D}\Lambda^{op})$ is the unique maximal n -orthogonal subcategory of $\text{mod } \Lambda$. In this case, we say that Λ admits a *trivial maximal n -orthogonal subcategory* of $\text{mod } \Lambda$ (see [HuZ]).

For a positive integer n , we proved in [HuZ, Proposition 3.2] that Λ admits no maximal j -orthogonal subcategories of $\text{mod } \Lambda$ for any $j \geq n$ if $\text{id}_\Lambda \Lambda = n$ (especially, if $\text{gl.dim } \Lambda = n$).

Furthermore, in [HuZ] we gave an equivalent characterization for the existence of trivial maximal $(n - 1)$ -orthogonal subcategories of $\text{mod } \Lambda$ over an $(n - 1)$ -Auslander algebra Λ with $\text{gl.dim } \Lambda = n$ as follows.

Lemma 3.7 ([HuZ, Corollary 3.10]) *Let Λ be an $(n - 1)$ -Auslander algebra with $\text{gl.dim } \Lambda = n$. Then the following statements are equivalent.*

- (1) Λ admits a trivial maximal $(n - 1)$ -orthogonal subcategory $\text{add}_\Lambda(\Lambda \oplus \mathbb{D}\Lambda^{op})$ of $\text{mod } \Lambda$.
- (2) A simple module $S \in \text{mod } \Lambda$ is injective if $\text{pd}_\Lambda S = n$.

For a positive integer n , recall from [FGR] that Λ is called *n-Gorenstein* if $\text{pd}_\Lambda I^i(\Lambda) \leq i$ for any $0 \leq i \leq n - 1$. By [FGR, Theorem 3.7], the notion of *n-Gorenstein* is left and right symmetric. Recall from [B] that Λ is called *Auslander-Gorenstein* if Λ is *n-Gorenstein* for all n and both $\text{id}_\Lambda \Lambda$ and $\text{id}_{\Lambda^{op}} \Lambda$ are finite.

Lemma 3.8 *Assume that $\text{id}_\Lambda \Lambda = \text{id}_{\Lambda^{op}} \Lambda = n (< \infty)$. Then we have the following*

- (1) ([IS, Proposition 1(1)]) $\text{pd}_\Lambda X = n$ or ∞ for any non-zero submodule X of $I^n(\Lambda)$.
- (2) ([IS, Corollary 7(2)]) If Λ is Auslander-Gorenstein and $I \in \mathcal{I} \mathcal{P}^n(\Lambda)$, then $I \cong I^0(S)$ for some simple module $S \in \text{mod } \Lambda$ with $\text{pd}_\Lambda S = n$ or ∞ .

For a module $M \in \text{mod } \Lambda$, the *grade* of M , denoted by $\text{grade } M$, is defined as $\inf\{n \geq 0 \mid \text{Ext}_\Lambda^n(M, \Lambda) \neq 0\}$ (see [AuB]).

Lemma 3.9 ([Iy1, Proposition 2.4]) *Let Λ be *n-Gorenstein*. Then the subcategory $\{X \in \text{mod } \Lambda \mid \text{grade } X \geq n\}$ of $\text{mod } \Lambda$ is closed under submodules and factor modules.*

Lemma 3.10 ([HuZ, Lemma 3.4]) *If $\text{gl.dim } \Lambda = n \geq 2$ and \mathcal{C} is a subcategory of $\text{mod } \Lambda$ such that $\Lambda \in \mathcal{C}$ and $\text{Ext}_\Lambda^j(\mathcal{C}, \mathcal{C}) = 0$ for any $1 \leq j \leq n - 1$, then $\text{grade } M = n$ for any $M \in \mathcal{C}$ without projective direct summands.*

4. The existence of trivial maximal orthogonal subcategories

In this section, we will mainly study the properties of $(n - 1)$ -Auslander algebras with $\text{gl.dim } \Lambda = n$ admitting a trivial maximal $(n - 1)$ -subcategory. We will prove that for an $(n - 1)$ -Auslander algebra Λ with $\text{gl.dim } \Lambda = n$, if Λ admits a trivial maximal $(n - 1)$ -orthogonal subcategory of $\text{mod } \Lambda$, then Λ is a Nakayama algebra and the projective dimension or injective dimension of any indecomposable module in $\text{mod } \Lambda$ is at most $n - 1$. As a consequence, we have that for an Auslander algebra Λ with $\text{gl.dim } \Lambda = 2$, if Λ admits a trivial maximal 1-orthogonal subcategory of $\text{mod } \Lambda$, then Λ is a tilted algebra of finite representation type.

Lemma 4.1 *Let Λ be an $(n-1)$ -Auslander algebra with $\text{gl.dim } \Lambda = n$ and $S \in \text{mod } \Lambda$ a simple module.*

(1) *If $\text{pd}_\Lambda S \leq n-1$, then $I^0(S)$ is projective.*

(2) *If $\text{pd}_\Lambda S = n$, then $\text{pd}_\Lambda I^0(S) = n$.*

Proof. For any $0 \leq i \leq n$, if $\text{pd}_\Lambda S = i$, then $\text{Hom}_\Lambda(S, I^i(\Lambda)) \cong \text{Ext}_\Lambda^i(S, \Lambda) \neq 0$. It follows that $I^0(S)$ is isomorphic to a direct summand of $I^i(\Lambda)$. Notice that Λ is an $(n-1)$ -Auslander algebra, then (1) follows trivially, and (2) follows from Lemma 3.8(1). \square

For a module $M \in \text{mod } \Lambda$, we use $\text{L}(M)$ to denote the length of M .

Lemma 4.2 *Let Λ be an $(n-1)$ -Auslander algebra with $\text{gl.dim } \Lambda = n$ admitting a trivial maximal $(n-1)$ -orthogonal subcategory of $\text{mod } \Lambda$ and $M \in \text{mod } \Lambda$ indecomposable. If $\text{L}(M) \geq 2$ or M is not injective, then the following equivalent conditions hold true.*

(1) *$\text{pd}_\Lambda S \leq n-1$ for any simple submodule S of M .*

(2) *$I^0(M)$ is projective.*

Proof. By Lemma 3.7, a simple module $S \in \text{mod } \Lambda$ is injective if $\text{pd}_\Lambda S = n$. Because $M \in \text{mod } \Lambda$ is indecomposable, we have that $\text{pd}_\Lambda S \leq n-1$ for any simple submodule S of M and the assertion (1) holds true. Otherwise, $M \cong S$, which contradicts with the assumption that $\text{L}(M) \geq 2$ or M is not injective.

It suffices to prove (1) \Rightarrow (2). By Lemma 4.1(1), it is easy to get the desired conclusion.

\square

The following proposition plays a crucial role in the proof of the main result in this paper.

Proposition 4.3 *Let Λ be an $(n-1)$ -Auslander algebra with $\text{gl.dim } \Lambda = n$ and $0 \leq k \leq n$. If Λ admits a trivial maximal $(n-1)$ -orthogonal subcategory of $\text{mod } \Lambda$, then for any indecomposable non-projective-injective module $M \in \text{mod } \Lambda$ with $\text{pd}_\Lambda M = n-k$, there exists a simple module $S \in \text{mod } \Lambda$ such that $\text{pd}_\Lambda S = n$ and $M \cong \Omega^k S$.*

Proof. For the case $k = 0$, it suffices to prove that $\text{L}(M) = 1$. Then M is simple and it is injective by Lemma 3.7. Thus the assertion follows.

Assume that $\text{L}(M) \geq 2$. By Lemma 4.2, $\text{pd}_\Lambda S \leq n-1$ for any simple submodule S of M and $I^0(M)$ is projective.

If M is injective, then $M \cong I^0(S)$ for some simple Λ -module S with $\text{pd}_\Lambda S = n$ by Lemma 3.8(2), which is a contradiction. Now assume that $\text{id}_\Lambda M \geq 1$. By Corollary 2.4,

$$0 \rightarrow P_n(M) \rightarrow \cdots \rightarrow P_1(M) \rightarrow P_0(M) \rightarrow I^0(M) \xrightarrow{\pi} I^0(M)/M \rightarrow 0$$

is a minimal projective resolution of $I^0(M)/M$ and $\text{pd}_\Lambda I^0(M)/M = n+1$, which contradicts with $\text{gl.dim } \Lambda = n$. So the case for $k = 0$ is proved.

For the case $k = n$, we have that M is projective. Then M is not injective by assumption. Because $\text{gl.dim } \Lambda = n$, $\text{id}_\Lambda M \leq n$. On the other hand, because Λ admits a trivial maximal $(n-1)$ -orthogonal subcategory of $\text{mod } \Lambda$, $\text{Ext}_\Lambda^j(\mathbb{D}\Lambda^{\text{op}}, \Lambda) = 0$ for any $1 \leq j \leq n-1$. Then it is not difficult to show that $\text{id}_\Lambda M = n$. By Lemma 3.3, there exists an indecomposable injective module $T \in \text{mod } \Lambda$ with $\text{pd}_\Lambda T = n$ such that $M \cong \Omega^n T$. By the above argument, T is simple.

Now assume that $1 \leq k \leq n-1$. Then $\text{pd}_\Lambda M = n-k \neq 0$. We claim that M is not injective. Otherwise, if M is injective, then the minimal projective resolution of M splits because $\text{Ext}_\Lambda^j(\mathbb{D}\Lambda^{\text{op}}, \Lambda) = 0$ for any $1 \leq j \leq n-1$. It follows that M is projective, which is a contradiction. The claim is proved. Then by Lemma 4.2, $\text{pd}_\Lambda S \leq n-1$ for any simple submodule S of M and $I^0(M)$ is projective. In the following, we will prove the assertion by induction on k .

If $k = 1$, then $\text{pd}_\Lambda M = n-1$. By Lemma 2.6 and Corollary 2.4, $\text{pd}_\Lambda I^0(M)/M = n$. So $I^0(M)/M \cong S$ for some simple module S with $\text{pd}_\Lambda S = n$ by the above argument and hence $M \cong \Omega^1 S$.

Assume that $2 \leq k \leq n-1$ and $\text{pd}_\Lambda M = n-k$. By Corollary 2.4, we have a minimal projective resolution of $I^0(M)/M$ as follows.

$$0 \rightarrow P_{n-k}(M) \rightarrow \cdots \rightarrow P_1(M) \rightarrow P_0(M) \rightarrow I^0(M) \xrightarrow{\pi} I^0(M)/M \rightarrow 0.$$

So $\text{pd}_\Lambda I^0(M)/M = n-(k-1)$ and $I^0(M)/M$ is indecomposable by Lemma 2.6. By the induction hypothesis, $I^0(M)/M \cong \Omega^{k-1} S$ for some simple module $S \in \text{mod } \Lambda$ with $\text{pd}_\Lambda S = n$. It follows that $M \cong \Omega^k S$. \square

As an application of Proposition 4.3, we get the following theorem, which is one of the main results in this section.

Theorem 4.4 *Let Λ be an $(n-1)$ -Auslander algebra with $\text{gl.dim } \Lambda = n$. If Λ admits a trivial maximal $(n-1)$ -orthogonal subcategory of $\text{mod } \Lambda$, then*

- (1) $\text{pd}_\Lambda M + \text{id}_\Lambda M = n$ for any non-projective-injective indecomposable module $M \in \text{mod } \Lambda$.
- (2) $\text{pd}_\Lambda M \leq n-1$ or $\text{id}_\Lambda M \leq n-1$ for any indecomposable module $M \in \text{mod } \Lambda$.

Proof. (1) Assume that $\text{pd}_\Lambda M = n-k$ for some $0 \leq k \leq n$. By Proposition 4.3, there exists a simple module $S \in \text{mod } \Lambda$ such that $\text{pd}_\Lambda S = n$ and $M \cong \Omega^k S$. Then S is injective

by Lemma 3.4 and so S is isomorphic to a direct summand of $\mathbb{D}\Lambda$. Because Λ is an $(n-1)$ -Auslander algebra, $P_i(S)$ is injective for any $0 \leq i \leq n-1$ by Lemma 3.2. Then, by Lemma 2.6, the following exact sequence:

$$0 \rightarrow \Omega^k S \rightarrow P_{k-1}(S) \rightarrow \cdots \rightarrow P_1(S) \rightarrow P_0(S) \rightarrow S \rightarrow 0$$

is a minimal injective resolution of $\Omega^k S (\cong M)$ and $\text{id}_\Lambda M = k$.

(2) follows from (1) immediately. \square

The following result is another application of Proposition 4.3.

Proposition 4.5 *Let Λ be an $(n-1)$ -Auslander algebra with $\text{gl.dim } \Lambda = n$ and $S \in \text{mod } \Lambda$ a simple module with $\text{pd}_\Lambda S = n$. If Λ admits a trivial maximal $(n-1)$ -orthogonal subcategory of $\text{mod } \Lambda$, then $\Omega^i S$ is simple and $P_i(S)$ is indecomposable for any $0 \leq i \leq n$.*

Proof. Assume that $S \in \text{mod } \Lambda$ is a simple module with $\text{pd}_\Lambda S = n$. By Lemma 3.7, S is injective. It follows from Lemma 3.2(2) that the minimal projective resolution of S :

$$0 \rightarrow P_n(S) \rightarrow \cdots \rightarrow P_1(S) \rightarrow P_0(S) \rightarrow S \rightarrow 0$$

is a minimal injective resolution of $P_n(S)$.

We proceed by induction on i . The case for $i = 0$ holds true trivially, and the case for $i = n$ follows from Lemma 3.4 and the dual version of Proposition 4.3.

Now assume that $1 \leq i \leq n-1$ and $S' \in \text{mod } \Lambda$ is a simple submodule of $\Omega^i S$. Because Λ is an $(n-1)$ -Auslander algebra and S is injective, $P_0(S)$ is projective-injective and indecomposable. So S' is the unique simple submodule of $P_0(S)$ and hence $I^0(S') = P_0(S)$. By Lemma 2.2, $\text{Ext}_\Lambda^{n-1}(S', P_n(S)) \cong \text{Hom}_\Lambda(S', \Omega^1 S) \neq 0$, which implies that $\text{pd}_\Lambda S' \geq n-1$. Because $\text{gl.dim } \Lambda = n$, it is easy to see that $\text{pd}_\Lambda S' = n-1$. Then by Theorem 4.4, $\text{id}_\Lambda S' = 1$.

Connecting a minimal projective resolution and a minimal injective resolution of S' , then by Lemma 2.6, the following exact sequence is a minimal projective resolution of $I^1(S')$:

$$0 \rightarrow P_{n-1}(S') \rightarrow \cdots \rightarrow P_0(S') \rightarrow I^0(S') (\cong P_0(S)) \rightarrow I^1(S') \rightarrow 0$$

with $I^1(S')$ indecomposable. So $\text{pd}_\Lambda I^1(S') = n$ and hence $I^1(S')$ is simple by Lemma 3.2(1). It follows that $S \cong I^1(S')$ and then $\Omega^1 S \cong S'$ is simple. Thus $P_1(S)$ is indecomposable. The case for $i = 1$ is proved.

Assume that $2 \leq i \leq n-1$. By Lemma 2.2, $\text{Ext}_\Lambda^{n-i}(S', P_n(S)) \cong \text{Hom}_\Lambda(S', \Omega^i S) \neq 0$. So $\text{pd}_\Lambda S' (= t) \geq n-i$.

Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & S' & \longrightarrow & P_{i-1}(S) & \xrightarrow{\pi} & M \longrightarrow 0 \\
& & \downarrow \alpha & & \parallel & & \downarrow \beta \\
0 & \longrightarrow & \Omega^i S & \longrightarrow & P_{i-1}(S) & \longrightarrow & \Omega^{i-1} S \longrightarrow 0
\end{array}$$

where $M = P_{i-1}(S)/S'$, α is an embedding homomorphism and β is an induced homomorphism. By the induction hypothesis, $\Omega^{i-1}S$ is simple and hence $P_{i-1}(S)$ is indecomposable. Then, by Lemma 2.6, M is indecomposable and π is right minimal. It follows that $\text{pd}_\Lambda M = t + 1$. Thus $M \cong \Omega^{n-t-1}S''$ for some simple module $S'' \in \text{mod } \Lambda$ with $\text{pd}_\Lambda S'' = n$ by Proposition 4.3. Because $i \geq n - t - 1$, M is simple by the induction hypothesis. It is clear that β is an epimorphism and so it is an isomorphism, which implies that α is an isomorphism and $\Omega^i S \cong S'$ is simple. It follows that $P_i(S)$ is indecomposable. \square

As an application of Proposition 4.5, we get the following

Corollary 4.6 *Let Λ be an $(n - 1)$ -Auslander algebra with $\text{gl.dim } \Lambda = n$. If Λ admits a trivial maximal $(n - 1)$ -orthogonal subcategory of $\text{mod } \Lambda$, then for any $0 \leq i \leq n$, Ω^i gives a bijection from $\{[S] \mid S \in \text{mod } \Lambda \text{ is simple with } \text{pd}_\Lambda S = n\}$ to $\{[S] \mid S \in \text{mod } \Lambda \text{ is non-projective-injective and simple with } \text{pd}_\Lambda S = n - i\}$, where $[S]$ is the isomorphic class of modules in $\text{mod } \Lambda$ containing S .*

Proof. By Proposition 4.5, $\Omega^i : \{[S] \mid S \in \text{mod } \Lambda \text{ is simple with } \text{pd}_\Lambda S = n\} \rightarrow \{[S] \mid S \in \text{mod } \Lambda \text{ is simple with } \text{pd}_\Lambda S = n - i\}$ is a map. By Proposition 4.3, Ω^i is epic. On the other hand, any simple module $S \in \text{mod } \Lambda$ with $\text{pd}_\Lambda S = n$ is injective by Lemmas 3.7 or Theorem 4.4, so the minimal projective resolution of S is a minimal injective resolution of $P_n(S)$ by Lemma 3.2(2). In particular, $0 \rightarrow \Omega^i S \rightarrow P_{i-1}(S) \rightarrow \cdots \rightarrow P_1(S) \rightarrow P_0(S) \rightarrow S \rightarrow 0$ is a minimal injective resolution of $\Omega^i S$. Then it is easy to see that Ω^i is monic. \square

We give another application of Proposition 4.5 as follows.

Corollary 4.7 *Let Λ be an $(n - 1)$ -Auslander algebra with $\text{gl.dim } \Lambda = n$. If Λ admits a trivial maximal $(n - 1)$ -orthogonal subcategory of $\text{mod } \Lambda$, then for any indecomposable projective module $P \in \text{mod } \Lambda$, either P or the radical $\text{rad } P$ of P is simple.*

Proof. Let $P \in \text{mod } \Lambda$ be an indecomposable projective module. Then there exists a unique (up to isomorphisms) simple module $S \in \text{mod } \Lambda$ such that $P \cong P_0(S)$. If S is projective, then P is simple. Now suppose $\text{pd}_\Lambda S = n - k > 0$. Then by Proposition 4.3, there exists a simple module $S' \in \text{mod } \Lambda$ with $\text{pd}_\Lambda S' = n$ such that $S \cong \Omega^k S'$. By Proposition 4.5, $\text{rad } P (\cong \Omega^1 S \cong \Omega^{k+1} S')$ is simple. \square

Definition 4.8 ([AuRS]) Λ is called a *Nakayama algebra* if every indecomposable projective module and every indecomposable injective module in $\text{mod } \Lambda$ have a unique composition series.

The following theorem is another main result in this section.

Theorem 4.9 *Let Λ be an $(n-1)$ -Auslander algebra with $\text{gl.dim } \Lambda = n$. If Λ admits a trivial maximal $(n-1)$ -orthogonal subcategory of $\text{mod } \Lambda$, then Λ is a Nakayama algebra.*

Proof. Let $P \in \text{mod } \Lambda$ be an indecomposable projective module. Then $\text{rad } P$ is the unique maximal submodule of P . It follows from Corollary 4.7 that P has a unique composition series of length at most two.

Note that Λ^{op} is also an $(n-1)$ -Auslander algebra admitting a trivial maximal $(n-1)$ -orthogonal subcategory of $\text{mod } \Lambda^{op}$. So by the above argument, every indecomposable projective module in $\text{mod } \Lambda^{op}$ has a unique composition series of length at most two. Applying the functor \mathbb{D} , then we get that every indecomposable injective module in $\text{mod } \Lambda$ has a unique composition series of length at most two. Thus Λ is a Nakayama algebra. \square

The following example illustrates that there exists a basic and connected $(n-1)$ -Auslander algebra Λ with $\text{gl.dim } \Lambda = n$, which is a Nakayama algebra, but admits no maximal $(n-1)$ -orthogonal subcategories of $\text{mod } \Lambda$. It means that the converse of Theorem 4.9 does not hold true in general.

Example 4.10 Let Λ be a finite-dimensional algebra over an algebraically closed field given by the quiver:

$$1 \xleftarrow{\beta_1} 2 \xleftarrow{\beta_2} 3 \xleftarrow{\beta_3} \cdots \xleftarrow{\beta_{n-1}} n \xleftarrow{\beta_n} n+1 \xleftarrow{\beta_{n+1}} n+2 \xleftarrow{\beta_{n+2}} \cdots \xleftarrow{\beta_{2n-2}} 2n-1 \xleftarrow{\beta_{2n-1}} 2n \xleftarrow{\beta_{2n}} 2n+1$$

modulo the ideal generated by $\{\beta_i \beta_{i+1} \mid 1 \leq i \leq 2n-1 \text{ but } i \neq n\}$. Then Λ is a basic and connected $(n-1)$ -Auslander algebra with $\text{gl.dim } \Lambda = n$. By [AsSS, Chapter V, Theorem 3.2] (see Lemma 5.2 below), Λ is a Nakayama algebra. We use $P(i)$, $I(i)$ and $S(i)$ to denote the projective, injective and simple modules corresponding to the vertex i for any $1 \leq i \leq 2n+1$. Because $P(n+2) = I(n)$ is not simple, it follows from [AsSS, Chapter IV, Proposition 3.11] that $0 \rightarrow P(n+1) \rightarrow S(n+1) \oplus P(n+2) \rightarrow I(n+1) \rightarrow 0$ is an almost split sequence. So $\text{Ext}_{\Lambda}^1(I(n+1), P(n+1)) \neq 0$ and hence there does not exist a maximal j -orthogonal subcategory of $\text{mod } \Lambda$ for any $j \geq 1$.

In the rest of this section, we will apply Theorems 4.4 and 4.9 to Auslander algebras with global dimension 2. As a result, we can give a connection between Auslander algebras and tilted algebras. We first recall some notions from [HRS] and [HRi].

Definition 4.11 (1) ([HRS]) Λ is called *almost hereditary* if the following conditions are satisfied: (a) $\text{gl.dim } \Lambda \leq 2$; and (b) If $X \in \text{mod } \Lambda$ is indecomposable, then either $\text{pd}_\Lambda X \leq 1$ or $\text{id}_\Lambda X \leq 1$.

(2) ([HRS]) Λ is called *quasi-tilted* if $\Lambda = \text{End}(T)^{\text{op}}$, where \mathcal{H} is a locally finite hereditary abelian R -category and T is a tilting object in \mathcal{H} .

(3) ([HRi]) Λ is called *tilted* if Λ is of the form $\Lambda = \text{End}(T_\Gamma)$, where T_Γ is a tilting module and Γ is a hereditary Artinian algebra. It is trivial that a tilted algebra is quasi-tilted.

Now we are in a position to give the following result.

Corollary 4.12 *Let Λ be an Auslander algebra with $\text{gl.dim } \Lambda = 2$. If Λ admits a trivial maximal 1-orthogonal subcategory of $\text{mod } \Lambda$, then Λ is a tilted algebra of finite representation type.*

Proof. Let Λ be an Auslander algebra with $\text{gl.dim } \Lambda = 2$ admitting a trivial maximal 1-orthogonal subcategory of $\text{mod } \Lambda$. Then Λ is an almost hereditary algebra of finite representation type by Theorems 4.4 and 4.9. So Λ is quasi-tilted by [HRS, Chapter III, Theorem 2.3] and hence it is tilted by [HRS, Chapter III, Corollary 3.6]. \square

Remark. (1) Let Λ be an Auslander algebra (of finite representation type) with $\text{gl.dim } \Lambda = 2$. If Λ admits a non-trivial maximal 1-orthogonal subcategory of $\text{mod } \Lambda$ (note: Iyama in [Iy5] constructed an example to illustrate that this may occur), then Λ is not almost hereditary because any maximal 1-orthogonal subcategory (if it exists) for an almost hereditary algebra is trivial by [HuZ, Theorem 3.15]. So Λ is not (quasi-)tilted by [HRS, Chapter III, Theorem 2.3].

(2) In the statement of Corollary 4.12, the conditions “ Λ is an Auslander algebra” and “ Λ is a tilted algebra of finite representation type” cannot be exchanged. For example, let Λ be a finite-dimensional algebra given by the quiver:

$$1 \xleftarrow{\alpha_1} 2 \xleftarrow{\alpha_2} 3 \xleftarrow{\alpha_3} 4 \xleftarrow{\alpha_4} 5$$

modulo the ideal generated by $\{\alpha_1\alpha_2\alpha_3\alpha_4\}$. Then Λ is a tilted algebra of finite representation type (cf. [AsSS, p.323]), and Λ admits a trivial maximal 1-orthogonal subcategory $\text{add}_\Lambda \bigoplus_{i=1}^5 P(i) \oplus I(3) \oplus I(4) \oplus I(5)$ of $\text{mod } \Lambda$. However, Λ is not an Auslander algebra because $\text{pd}_\Lambda I^1(\Lambda) = 2$.

5. The case for finite-dimensional algebras

In this section, Λ is a finite-dimensional algebra over an algebraically closed field K . As

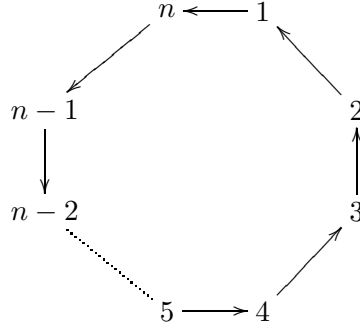
an application of the results obtained in Section 4, we will give an explicit description of the quiver of an $(n-1)$ -Auslander algebra Λ with $\text{gl.dim } \Lambda = n$ admitting a trivial maximal $(n-1)$ -orthogonal subcategory of $\text{mod } \Lambda$. We begin with some preliminary results.

Lemma 5.1 *Let $\{P_1, P_2, \dots, P_m\}$ be a complete set of non-isomorphic indecomposable projective modules in $\text{mod } \Lambda$. Then Λ is connected if and only if there does not exist a non-trivial partition $J_1 \dot{\cup} J_2$ of the set $\{P_1, P_2, \dots, P_m\}$ such that $P_i \in J_1$ and $P_j \in J_2$ imply $\text{Hom}_\Lambda(P_i, P_j) = 0 = \text{Hom}_\Lambda(P_j, P_i)$.*

Proof. By [AsSS, Chapter II, Lemma 1.6]. □

Lemma 5.2 ([AsSS, Chapter V, Theorem 3.2]) *Let Λ be basic and connected. Then Λ is a Nakayama algebra if and only if its ordinary quiver is one of the following two quivers:*

- (1) $1 \leftarrow 2 \leftarrow 3 \leftarrow \dots \leftarrow n-1 \leftarrow n$;
- (2)



(with $n \geq 1$ points).

The following proposition is useful for proving the main result in this section.

Proposition 5.3 *Let Λ be a basic and connected $(n-1)$ -Auslander algebra with $\text{gl.dim } \Lambda = n$. If Λ admits a trivial maximal $(n-1)$ -orthogonal subcategory of $\text{mod } \Lambda$, then there exists a unique simple module $S \in \text{mod } \Lambda$ with $\text{pd}_\Lambda S = n$.*

Proof. Since Λ is connected, it is not difficult to see that there does not exist a simple projective-injective module in $\text{mod } \Lambda$ by Lemma 5.1. Then by Corollary 4.6, every simple module in $\text{mod } \Lambda$ is of the form $\Omega^i S$ for some simple module $S \in \text{mod } \Lambda$ with $\text{pd}_\Lambda S = n$ and $0 \leq i \leq n$. Because $\text{gl.dim } \Lambda = n$, there exists a simple module $S \in \text{mod } \Lambda$ with $\text{pd}_\Lambda S = n$.

Assume that $\{S_1, S_2, \dots, S_t\}$ is a complete set of non-isomorphic simple modules in $\text{mod } \Lambda$ with projective dimension n . It suffices to prove $t = 1$. Suppose $t \geq 2$. By Corollary 4.6, we get a complete set of non-isomorphic indecomposable projective modules in $\text{mod } \Lambda$

as follows.

$$\begin{aligned} & \{P_0(S_1), P_1(S_1)(= P_0(\Omega^1 S_1)), P_2(S_1)(= P_0(\Omega^2 S_1)), \dots, P_n(S_1)(= P_0(\Omega^n S_1)) \\ & P_0(S_2), P_1(S_2)(= P_0(\Omega^1 S_2)), P_2(S_2)(= P_0(\Omega^2 S_2)), \dots, P_n(S_2)(= P_0(\Omega^n S_2)) \\ & \dots\dots\dots \\ & P_0(S_t), P_1(S_t)(= P_0(\Omega^1 S_t)), P_2(S_t)(= P_0(\Omega^2 S_t)), \dots, P_n(S_t)(= P_0(\Omega^n S_t))\}. \end{aligned}$$

In the following, we will show

$$\text{Hom}_\Lambda(P_i(S_j), P_k(S_l)) = 0 = \text{Hom}_\Lambda(P_k(S_l), P_i(S_j)) \quad (*)$$

for any $0 \leq i, k \leq n$ and $1 \leq j \neq l \leq t$.

Notice that S_i is injective for any $1 \leq i \leq t$ by Lemma 3.7, then it follows from Lemma 3.2(2) that the minimal projective resolution of S_l :

$$0 \rightarrow P_n(S_l) \rightarrow \dots \rightarrow P_1(S_l) \rightarrow P_0(S_l) \rightarrow S_l \rightarrow 0 \quad (**)$$

is a minimal injective resolution of $P_n(S_l)$. We split (**) to the following n short exact sequences:

$$0 \rightarrow P_n(S_l) \rightarrow P_{n-1}(S_l) \rightarrow \Omega^{n-1} S_l \rightarrow 0 \quad (1)$$

.....

$$0 \rightarrow \Omega^2 S_l \rightarrow P_1(S_l) \rightarrow \Omega^1 S_l \rightarrow 0 \quad (n-1)$$

$$0 \rightarrow \Omega^1 S_l \rightarrow P_0(S_l) \rightarrow S_l \rightarrow 0 \quad (n)$$

Since $S_j \not\cong S_l$ and $\Omega^i S_j$ and $\Omega^i S_l$ are simple for any $0 \leq i \leq n$ by Proposition 4.5, we get that $\Omega^i S_j \not\cong \Omega^i S_l$ for any $0 \leq i \leq n$ by Corollary 4.6 (note: $\Omega^n S_j \cong P_n(S_j)$ and $\Omega^n S_l \cong P_n(S_l)$). So $\Omega^i S_j \not\cong \Omega^k S_l$ for any $0 \leq i, k \leq n$.

By applying the functor $\text{Hom}_\Lambda(P_i(S_j), -)$ (where $0 \leq i \leq n$) to the exact sequences (1), \dots , $(n-1)$, (n) , then we get the following exact sequences:

$$0 = \text{Hom}_\Lambda(P_i(S_j), \Omega^{k+1} S_l) \rightarrow \text{Hom}_\Lambda(P_i(S_j), P_k(S_l)) \rightarrow \text{Hom}_\Lambda(P_i(S_j), \Omega^k S_l) = 0$$

for $k = 0, 1, \dots, n-1$. So $\text{Hom}_\Lambda(P_i(S_j), P_k(S_l)) = 0$ for any $0 \leq i \leq n-1$. In addition, it is trivial that $\text{Hom}_\Lambda(P_i(S_j), P_n(S_l)) = 0$. This proves the left equality of (*). Dually, we get the right equality of (*). Thus we get a non-trivial partition $J_1 \dot{\bigcup} J_2$ of the set of non-isomorphic indecomposable projective modules in $\text{mod } \Lambda$, where $J_1 = \{P_0(S_1), P_1(S_1), \dots, P_n(S_1)\}$ and

$J_2 = \{P_0(S_2), P_1(S_2), \dots, P_n(S_2), \dots, P_0(S_t), P_1(S_t), \dots, P_n(S_t)\}$, which is a contradiction by Lemma 5.1. The proof is finished. \square

Now we are in a position to give the main result in this section.

Theorem 5.4 *Λ is a basic and connected $(n-1)$ -Auslander algebra with $\text{gl.dim } \Lambda = n$ admitting a trivial maximal $(n-1)$ -orthogonal subcategory of $\text{mod } \Lambda$ if and only if Λ is given by the quiver:*

$$1 \xleftarrow{\beta_1} 2 \xleftarrow{\beta_2} 3 \xleftarrow{\beta_3} \cdots \xleftarrow{\beta_n} n+1$$

modulo the ideal generated by $\{\beta_i \beta_{i+1} | 1 \leq i \leq n-1\}$.

Proof. We first prove the sufficiency. It is straightforward to verify that Λ is an $(n-1)$ -Auslander algebra with $\text{gl.dim } \Lambda = n$ and admits a maximal $(n-1)$ -orthogonal subcategory $\mathcal{C} = \text{add}_\Lambda (P(1) \oplus P(2) \oplus P(3) \oplus \cdots \oplus P(n+1) \oplus S(n+1))$.

We then prove the necessity. By Theorem 4.9, Λ is a Nakayama algebra. By Proposition 5.3 and Corollary 4.6, there exist exactly $n+1$ non-isomorphic simple modules in $\text{mod } \Lambda$. Because $\text{gl.dim } \Lambda = n$, there exists a simple module $S \in \text{mod } \Lambda$ with $\text{pd}_\Lambda S = n$. By Lemma 3.7, S is injective. That is, there exists a simple injective module in $\text{mod } \Lambda$. Then by Lemma 5.2, the ordinary quiver of Λ is given by

$$1 \xleftarrow{\beta_1} 2 \xleftarrow{\beta_2} 3 \xleftarrow{\beta_3} \cdots \xleftarrow{\beta_n} n+1 .$$

We claim that $\beta_i \beta_{i+1} = 0$ for any $1 \leq i \leq n-1$. Otherwise, if $\beta_i \beta_{i+1} \neq 0$ for some $1 \leq i \leq n-1$, then neither $P(i+2)$ nor $\text{rad } P(i+2) (\cong \Omega^1 S(i+2))$ are simple, which is a contradiction by Corollary 4.7. The claim is proved. Because Λ is connected, the ideal generated by $\{\beta_i \beta_{i+1} | 1 \leq i \leq n-1\}$ is exactly the non-zero admissible ideal of KQ , and the assertion follows. \square

In the following, as an application of Theorem 5.4, we will establish the structure theorem of an $(n-1)$ -Auslander algebra Λ admitting a trivial maximal $(n-1)$ -orthogonal subcategory of $\text{mod } \Lambda$.

Definition 5.5 ([AuRS]) Let $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n$ be a product of indecomposable algebras and $1 = e_1 + \cdots + e_n$ the corresponding decomposition of the identity element 1 of Λ . Then the Λ_i are called the *blocks* of Λ .

Lemma 5.6 ([A, Chapter IV, Proposition 3]) *Let S and T be simple modules in $\text{mod } \Lambda$. Then the following statements are equivalent.*

- (1) *S and T lie in the same block.*

(2) There exist simple modules $S = S_1, S_2, \dots, S_m = T$ in $\text{mod } \Lambda$ such that for any $1 \leq i \leq m-1$, S_i and S_{i+1} are composition factors of an indecomposable projective module in $\text{mod } \Lambda$.

(3) There exist simple modules $S = S_1, S_2, \dots, S_m = T$ in $\text{mod } \Lambda$ such that for any $1 \leq i \leq m-1$, either $S_i = S_{i+1}$ or there exists a non-split extension of one of them by the other.

We give some elementary properties of modules in a block as follows.

Lemma 5.7 *Let Λ_i be a block of Λ and $M \in \text{mod } \Lambda_i$. Then we have*

- (1) *The submodules, quotient modules and finite direct sums of M are also in $\text{mod } \Lambda_i$.*
- (2) *M is simple in $\text{mod } \Lambda_i$ if and only if it is simple in $\text{mod } \Lambda$.*
- (3) *M is projective (resp. injective) in $\text{mod } \Lambda_i$ if and only if it is projective (resp. injective) in $\text{mod } \Lambda$.*
- (4) *$P_0(M)$ (resp. $I^0(M)$) $\in \text{mod } \Lambda_i$.*
- (5) *If Λ_j is a block of Λ with $j \neq i$, then $\text{Ext}_\Lambda^t(N, M) = 0$ for any $N \in \text{mod } \Lambda_j$ and $t \geq 0$.*

Proof. (1) It follows from [A, p.93].

(2) If M is simple in $\text{mod } \Lambda_i$, then by (1), the submodules of M as Λ -modules are in $\text{mod } \Lambda_i$, which implies that M is simple in $\text{mod } \Lambda$. The converse is trivial.

(3) If M is projective in $\text{mod } \Lambda$, then $M \oplus P \cong \Lambda^{(s)}$ for some projective module $P \in \text{mod } \Lambda$ and a positive integer s . It follows that $M \oplus \Lambda_i P \cong \Lambda_i M \oplus \Lambda_i P \cong \Lambda_i \Lambda^{(s)} \cong \Lambda_i^{(s)}$ and M is projective in $\text{mod } \Lambda_i$. The converse is trivial. By the usual duality \mathbb{D} , we get another assertion.

(4) From the exact sequence $P_0(M) \rightarrow M \rightarrow 0$ in $\text{mod } \Lambda$, we get an exact sequence $\Lambda_i P_0(M) \rightarrow \Lambda_i M (\cong M) \rightarrow 0$ in $\text{mod } \Lambda_i$. Notice that $P_0(M)$ is a projective cover of M and $0 \neq \Lambda_i P_0(M)$ is isomorphic to a direct summand of $P_0(M)$, so we have that $P_0(M) \cong \Lambda_i P_0(M) \in \text{mod } \Lambda_i$. By the usual duality \mathbb{D} , we get that $I^0(M) \in \text{mod } \Lambda_i$.

(5) If Λ_j is a block of Λ with $j \neq i$, then it follows from [A, p.93] that $\text{Hom}_\Lambda(N, M) = 0$ for any $N \in \text{mod } \Lambda_j$ and $M \in \text{mod } \Lambda_i$. From this fact the case for $t = 0$ follows. Then by applying the functor $\text{Hom}_\Lambda(-, M)$ to a minimal projective resolution of N in $\text{mod } \Lambda_j$, we get the assertion inductively. \square

By the following lemma, we can draw the Auslander-Reiten quiver of an $(n-1)$ -Auslander algebra Λ admitting a trivial maximal $(n-1)$ -orthogonal subcategory of $\text{mod } \Lambda$.

Lemma 5.8 *Let Λ be an $(n-1)$ -Auslander algebra with $\text{gl.dim } \Lambda = n$ and $S \in \text{mod } \Lambda$ a simple module with $\text{pd}_\Lambda S = n$. If Λ admits a trivial maximal $(n-1)$ -orthogonal subcategory*

of $\text{mod } \Lambda$, then

$$0 \rightarrow \Omega^{i+1}S \rightarrow P_{i-1}(S) \xrightarrow{\pi_i} \Omega^i S \rightarrow 0 \quad (*i)$$

is an almost split sequence for any $0 \leq i \leq n-1$.

Proof. It is obvious that $(*i)$ does not split. It suffices to prove that there exists a homomorphism $g : M \rightarrow P_{i-1}(S)$ such that $\pi_i g = f$ whenever $f : M \rightarrow \Omega^i S$ is not a split epimorphism with $M \in \text{mod } \Lambda$ indecomposable.

If M is projective, then the assertion follows trivially. If $1 \leq \text{pd}_\Lambda M = j \leq n$, then by Proposition 4.3, $M \cong \Omega^{n-j}S'$ for some simple module $S' \in \text{mod } \Lambda$ with $\text{pd}_\Lambda S' = n$. So $M(\cong \Omega^{n-j}S')$ and $\Omega^i S$ are simple by Proposition 4.5. Because $f : M \rightarrow \Omega^i S$ is not a split epimorphism, $f = 0$. Thus $g = 0$ is desired. \square

Now we state the structure theorem of an $(n-1)$ -Auslander algebra Λ admitting a trivial maximal $(n-1)$ -orthogonal subcategory of $\text{mod } \Lambda$ as follows.

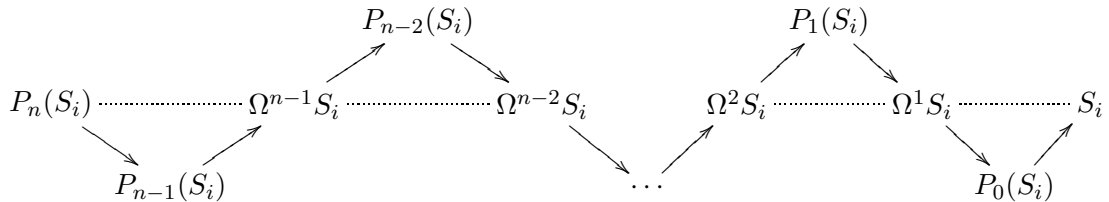
Theorem 5.9 *Λ is an $(n-1)$ -Auslander algebra Λ with $\text{gl.dim } \Lambda = n$ admitting a trivial maximal $(n-1)$ -orthogonal subcategory of $\text{mod } \Lambda$ if and only if it is a finite direct product of K and the algebra given by the quiver*

$$1 \xleftarrow{\beta_1} 2 \xleftarrow{\beta_2} 3 \xleftarrow{\beta_3} \cdots \xleftarrow{\beta_n} n+1$$

modulo the ideal generated by $\{\beta_i \beta_{i+1} | 1 \leq i \leq n-1\}$.

Proof. The sufficiency follows from Theorem 5.4 and Lemma 5.7. In the following, we will prove the necessity.

Let $\{S_1, S_2, \dots, S_t\} (t \geq 1)$ be a complete set of non-isomorphic simple modules in $\text{mod } \Lambda$ with projective dimension n . By Corollary 4.7, it is easy to see that for any $S_i, S_j \in \{S_1, S_2, \dots, S_t\}$ with $i \neq j$, there does not exist an indecomposable projective module $P \in \text{mod } \Lambda$ such that S_i and S_j are composition factors of P . It follows from Lemma 5.6 that S_i and S_j do not lie in the same block of Λ . Then we may assume that $\Lambda = K \times \cdots \times K \times \Lambda_1 \times \cdots \times \Lambda_t$ is a decomposition of blocks of Λ with $S_i \in \Lambda_i$ for any $S_i \in \{S_1, S_2, \dots, S_t\}$. By Proposition 4.3 and Lemma 5.8, for any $S_i \in \{S_1, S_2, \dots, S_t\}$, we get a connected component of the Auslander-Reiten quiver of Λ as follows.



$(**i)$

By Lemmas 5.6 and 5.7, we have that all modules lie in $(**i)$ are indecomposable modules in $\text{mod } \Lambda_i$, where $1 \leq i \leq t$. When $i \neq j$, because S_i and S_j do not lie in the same block, $(**i)$ and $(**j)$ do not lie in the same block. In the following, we will prove that all the indecomposable modules in $\text{mod } \Lambda_i$ lie in $(**i)$.

Let $0 \neq M \in \text{mod } \Lambda_i$ be indecomposable. Obviously, $M \in \text{mod } \Lambda$.

If M is non-projective-injective with $\text{pd}_\Lambda M = k$, then by Proposition 4.8, $M \cong \Omega^{n-k} S_j$ for some $S_j \in \{S_1, S_2, \dots, S_t\}$. So $0 \neq M \cong \Lambda_i M \cong \Lambda_i \Omega^{n-k} S_j$ and hence $j = i$ and $M \cong \Lambda_i \Omega^{n-k} S_i \cong \Omega^{n-k} S_i$, which lies in $(**i)$. If M is projective-injective, then we claim that $M \cong P_k(S_i)$ for some $0 \leq k \leq n-1$. Since Λ_i is a block, there does not exist a projective-injective simple module in $\text{mod } \Lambda_i$. Then by Lemma 5.7 and Corollary 4.6, for any simple module $S \in \text{mod } \Lambda_i$, $S \cong \Omega^k S_i$ for some $0 \leq k \leq n$. Since M is indecomposable injective, $M \cong I^0(S')$ for a simple module S' in $\text{mod } \Lambda_i$. So $S' \cong \Omega^k S_i$ for some $1 \leq k \leq n$, and hence by Lemma 2.6 $M \cong P_{k-1}(S_i)$ with $0 \leq k-1 \leq n-1$. The claim is proved. Thus M lies in $(**i)$.

Consequently, we conclude that Λ_i is a Nakayama algebra with $\text{gl.dim } \Lambda_i = n$ given by the quiver:

$$1 \xleftarrow{\beta_1} 2 \xleftarrow{\beta_2} 3 \xleftarrow{\beta_3} \cdots \xleftarrow{\beta_n} n+1$$

modulo the ideal generated by $\{\beta_i \beta_{i+1} | 1 \leq i \leq n-1\}$. The proof is finished. \square

6. Non-trivial maximal 1-orthogonal subcategories

In this section, based on [HuZ, Corollary 3.12], we will further give a necessary condition for Auslander algebras with global dimension 2 admitting a non-trivial maximal 1-orthogonal subcategory in terms of the homological properties of simple modules.

Lemma 6.1 *Let Λ be an Auslander algebra with $\text{gl.dim } \Lambda = 2$ and $S \in \text{mod } \Lambda$ a simple module with $\text{id}_\Lambda S = 2$. Then $I^2(S)$ is indecomposable and $I^0(S) \not\cong I^1(S)$.*

Proof. By Lemma 3.5, we get a simple module $S' \in \text{mod } \Lambda$ such that $\text{pd}_\Lambda S' = 2$ and $\mathbb{D} \circ \text{Ext}_\Lambda^2(S', \Lambda) = S$. From the minimal projective resolution of S' , we get an exact sequence:

$$0 \rightarrow P_0(S')^* \rightarrow P_1(S')^* \rightarrow P_2(S')^* \rightarrow \text{Ext}_\Lambda^2(S', \Lambda) \rightarrow 0,$$

which is a minimal projective resolution of $\text{Ext}_\Lambda^2(S', \Lambda)$ by [M, Proposition 4.2], where $(-)^* = \text{Hom}_\Lambda(-, \Lambda)$. Then Applying the functor \mathbb{D} , we get a minimal injective resolution of $S = \mathbb{D} \circ \text{Ext}_\Lambda^2(S', \Lambda)$:

$$0 \rightarrow S \rightarrow \mathbb{D}P_2(S')^* \rightarrow \mathbb{D}P_1(S')^* \rightarrow \mathbb{D}P_0(S')^* \rightarrow 0.$$

It follows that $I^2(S) \cong \mathbb{D}P_0(S')^*$, $I^1(S) \cong \mathbb{D}P_1(S')^*$ and $I^0(S) \cong \mathbb{D}P_2(S')^*$. On the other hand, from the minimal projective resolution of S' :

$$0 \rightarrow P_2(S') \rightarrow P_1(S') \rightarrow P_0(S') \rightarrow S' \rightarrow 0,$$

we know that $P_0(S')$ is indecomposable and $P_2(S') \not\cong P_1(S')$. So our assertion follows. \square

Proposition 6.2 *Let Λ be an Auslander algebra with $\text{gl.dim } \Lambda = 2$. If Λ admits a non-trivial maximal 1-orthogonal subcategory of $\text{mod } \Lambda$, then*

- (1) *There exists a simple module in $\text{mod } \Lambda$ with both projective and injective dimensions 2.*
- (2) *There exist at least two non-injective simple modules in $\text{mod } \Lambda$ with projective dimension 2.*

Proof. (1) It follows from [HuZ, Corollary 3.12].

(2) By (1), there exists a non-injective simple module in $\text{mod } \Lambda$ with projective dimension 2. If the non-injective simple module in $\text{mod } \Lambda$ with projective dimension 2 is unique (say S), then $\text{id}_\Lambda S = 2$ by (1). Since $I^0(S)$ and $I^2(S)$ are indecomposable by Lemma 6.1, $\text{grade } I^0(S) = \text{grade } I^2(S) = 2$ by Lemma 3.10. Put $K = \text{Coker}(S \hookrightarrow I^0(S))$. Then $\text{grade } K = 2$ by Lemma 3.9 and so $\text{grade } I^1(S) = 2$. We claim that $I^0(S)$ is isomorphic to a direct summand of $I^1(S)$. Otherwise, since S is the unique non-injective simple module with projective dimension 2, any non-zero indecomposable direct summand of $I^1(S)$ is simple by Lemma 3.8(2). So $I^1(S)$ is semisimple and hence K is injective, which contradicts with $\text{id}_\Lambda S = 2$.

Notice that $I^2(S)$ is indecomposable and $\text{pd}_\Lambda I^2(S) = 2$, so $I^2(S) \cong I^0(S)$ or $I^2(S) \cong S'$ for some simple module $S' \in \text{mod } \Lambda$ such that $S \not\cong S'$ and $\text{pd}_\Lambda S' = 2$. In the latter case, we have that $\mathbb{L}(I^0(S)) = \mathbb{L}(I^1(S))$. Since $I^0(S)$ is isomorphic to a direct summand of $I^1(S)$ by the above argument, $I^0(S) \cong I^1(S)$, which is a contradiction by Lemma 6.1.

Because Λ is an Auslander algebra and $\text{pd}_{\Lambda^{op}} \mathbb{D}S = 2$, it follows from Lemma 3.10 that $\text{grade } \mathbb{D}S = 2$. Then, for any injective module $I \in \text{mod } \Lambda$, $\text{Ext}_\Lambda^1(I, S) \cong \text{Ext}_{\Lambda^{op}}^1(\mathbb{D}S, \mathbb{D}I) = 0$. Moreover, $S \hookrightarrow I^0(S)$ is left minimal, thus K has no injective direct summands by Lemma 2.5 and therefore K is indecomposable by Lemmas 6.1 and 2.1. It follows from Lemma 2.3 that $I^1(S) \rightarrow I^2(S)$ is right minimal. So, if $I^2(S) \cong I^0(S)$, then $I^1(S)$ has no simple direct summand S'' such that $S'' \not\cong S$ and $\text{pd}_\Lambda S'' = 2$. It yields that $I^1(S) \cong [I^0(S)]^t$ for some $t \geq 1$ and $2\mathbb{L}(I^0(S)) = t\mathbb{L}(I^0(S)) + 1$. It implies that $t = 1$ and $I^0(S) \cong I^1(S)$, which is a contradiction by Lemma 6.1. The proof is finished. \square

As an immediate consequence of Proposition 6.2, we have the following result, which

gives some sufficient conditions that any maximal 1-orthogonal subcategory of $\text{mod } \Lambda$ (in case it exists) is trivial for an Auslander algebra Λ .

Corollary 6.3 *Let Λ be an Auslander algebra with $\text{gl.dim } \Lambda = 2$. Then any maximal 1-orthogonal subcategory of $\text{mod } \Lambda$ (in case it exists) is trivial if one of the following conditions are satisfied.*

- (1) *There exists a unique simple module with projective dimension 2.*
- (2) *There exist exactly two simple modules with projective dimension 2 and at least one of them is injective.*

From the results obtained in this paper and in [HuZ], we see that for an $(n-1)$ -Auslander algebra Λ with $\text{gl.dim } \Lambda = n$, the properties of simple modules with projective dimension n play an important role in the study of the existence of maximal $(n-1)$ -orthogonal subcategories and the properties of Λ admitting maximal $(n-1)$ -orthogonal subcategories.

We end this section with examples to illustrate Proposition 6.2 and Corollary 6.3.

The following example shows that there exists an Auslander algebra Λ with $\text{gl.dim } \Lambda = 2$ satisfying the condition (1) in Proposition 6.2, but not satisfying the condition (2) in this proposition.

Example 6.4 Let Λ be a finite-dimensional algebra given by the quiver:

$$1 \xleftarrow{\beta_1} 2 \xleftarrow{\beta_2} 3 \xleftarrow{\beta_3} 4 \xleftarrow{\beta_4} 5$$

modulo the ideal generated by $\{\beta_1\beta_2, \beta_3\beta_4\}$. Then we have

- (1) Λ is an Auslander algebra with $\text{gl.dim } \Lambda = 2$.
- (2) All simple modules in $\text{mod } \Lambda$ with projective dimension 2 are $S(3)$ and $S(5)$.
- (3) $\text{id}_\Lambda S(3) = 2$ and $S(5)$ is injective.

Then by Lemma 3.7 and Proposition 6.2(2), there does not exist any maximal 1-orthogonal subcategory of $\text{mod } \Lambda$.

The following example shows that there exists an Auslander algebra Λ with $\text{gl.dim } \Lambda = 2$ satisfying the condition (2) in Proposition 6.2, but not satisfying the condition (1) in this proposition.

Example 6.5 Let Λ be a finite-dimensional algebra given by the quiver:

$$\begin{array}{ccccc} 6 & \xrightarrow{\alpha} & 4 & & \\ \downarrow \gamma & & \downarrow \beta & & \\ 5 & \xrightarrow{\delta} & 3 & \xrightarrow{\lambda} & 1 \\ & & \downarrow \mu & & \\ & & 2 & & \end{array}$$

modulo the ideal generated by $\{\beta\alpha - \delta\gamma, \mu\delta, \lambda\beta\}$. Then we have

- (1) Λ is an Auslander algebra and an almost hereditary algebra with $\text{gl.dim } \Lambda = 2$.
- (2) All simple modules in $\text{mod } \Lambda$ with projective dimension 2 are $S(4)$, $S(5)$ and $S(6)$.
- (3) $\text{id}_\Lambda S(4) = \text{id}_\Lambda S(5) = 1$ and $S(6)$ is injective.

Then by Lemma 3.7 and Proposition 6.2(1), there does not exist any maximal 1-orthogonal subcategory of $\text{mod } \Lambda$.

According to Examples 6.4 and 6.5, we know that the conditions (1) and (2) in Proposition 6.2 are independent.

The following example is also related to Proposition 6.2 and Corollary 6.3. It shows that there exists an Auslander algebra Λ with $\text{gl.dim } \Lambda = 2$ and there exists a unique simple module $S \in \text{mod } \Lambda$ with $\text{pd}_\Lambda S = 2$ and $\text{id}_\Lambda S = 2$.

Example 6.6 Let Λ be a finite-dimensional algebra given by the quiver:

$$1 \begin{matrix} \xrightarrow{\beta} \\ \xleftarrow{\alpha} \end{matrix} 2$$

modulo the ideal generated by $\beta\alpha$. Then Λ is an Auslander algebra with $\text{gl.dim } \Lambda = 2$ and the unique simple module with projective dimension 2 is $S(2)$, and $\text{id}_\Lambda S(2) = 2$. Then by Proposition 6.2(2) (or Corollary 6.3) and Lemma 3.7, there does not exist any maximal 1-orthogonal subcategory of $\text{mod } \Lambda$.

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